Quantum mechanical potentials related to the prime numbers and Riemann zeros

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(Received 8 June 2008; published 25 November 2008)

Prime numbers are the building blocks of our arithmetic; however, their distribution still poses fundamental questions. Riemann showed that the distribution of primes could be given explicitly if one knew the distribution of the nontrivial zeros of the Riemann $\zeta(s)$ function. According to the Hilbert-Pólya conjecture, there exists a Hermitian operator of which the eigenvalues coincide with the real parts of the nontrivial zeros of $\zeta(s)$. This idea has encouraged physicists to examine the properties of such possible operators, and they have found interesting connections between the distribution of zeros and the distribution of energy eigenvalues of quantum systems. We apply the Marčhenko approach to construct potentials with energy eigenvalues equal to the prime numbers and to the zeros of the $\zeta(s)$ function. We demonstrate the multifractal nature of these potentials by measuring the Rényi dimension of their graphs. Our results offer hope for further analytical progress.

DOI: 10.1103/PhysRevE.78.056215 PACS number(s): 05.45.Mt, 02.10.De, 02.30.Zz, 03.65.Ge

I. PRIMES, ZEROS, AND QUANTA

The prime numbers are the building blocks for the positive integers, since the fundamental theorem of arithmetic states that every positive integer can be written as a product of primes, and this product is unique up to a rearrangement of the factors. Additionally, not only does the product of prime numbers have this remarkable property, but also their sum. Only a decade ago Ramaré proved [1] that any positive integer can be written as a sum of no more than six prime numbers, but the Goldbach conjecture [2], that every number is expressible as a sum of two prime numbers, remains unproven.

Based on empirical evidence, many mathematicians conjectured that the prime counting function, $\pi(x) = |\{p | p \text{ is prime and } p \leqslant x\}|$, asymptotically behaves as the logarithmic integral $\mathrm{Li}(x)$. Hadamard [3] and de la Vallée-Poussin [4] independently gave a rigorous proof for this statement. Riemann derived the following exact formula [5]:

$$\pi(x) = \lim_{x\to\infty}\left(R(x) - \sum_{\rho} R(x^{\rho})\right), \tag{1}$$

where $R(x)$ is the so-called Riemann function defined as $R(x)=\sum_{m=1}^{\infty}\mu(m)\mathrm{Li}(x^{1/m})/m$, and $\mu(m)$ denotes the Möbius function. The sum in (1) is extended over all nontrivial zeros ρ of the Riemann $\zeta(s)$ function, counted with their multiplicities. The latter function $\zeta(s)$ is defined by the infinite series $\zeta(s)=\sum_{n=1}^{\infty}n^{-s}$ for $s>1$, and, otherwise, by its analytic continuation over the complex s plane.

Exploring the locations of the zeros of $\zeta(s)$, Riemann made his famous conjecture: all the nontrivial zeros lie on the $s=\frac{1}{2}+it$ (t is real) critical line. Proof or disproof of the Riemann hypothesis remains the most tantalizing challenge in number theory since Hilbert nominated it in 1900 [6] as the eighth problem on his famous list of compelling problems in mathematics [7].

The connection between the Riemann hypothesis and physics seems to date back to the early years of quantum mechanics. According to the Hilbert-Pólya conjecture, the zeros of $\zeta(s)$ can be the spectrum of an operator $\mathcal{O}=\frac{1}{2}\mathcal{I}+i\mathcal{H}$, where $\mathcal{H}$ is self-adjoint. This operator $\mathcal{H}$ might have a physical interpretation as a Hamiltonian of a physical system and, therefore, the key to the proof of the Riemann hypothesis may have been coded in physics.

The analogy between the properties of $\zeta(s)$ and the energy eigenvalues of a quantum mechanical system provide us some information about the form of a possible operator $\mathcal{H}$ [8]. One of these similarities, the comparison of the number of $\zeta(s)$ zeros and the number of energy eigenvalues below a threshold, suggests that the physical system is quasi-one-dimensional. This link is further strengthened by checking different statistics of the zeros, such as the nearest-neighbor spacings, the n-correlations between the zeros, etc. Montgomery showed that these distributions are all in good agreement with the Gaussian unitary ensemble of random matrix theory [9].

In this work we utilize an inverse scattering formalism and construct potentials of which the energy eigenvalues are the zeros of the Riemann $\zeta(s)$ function. We also consider the problem when the eigenvalues are taken to be the prime numbers themselves. In Sec. II we introduce our numerical method and give evidence that it is capable of generating potentials from sets of discrete energy eigenvalues, such as a finite set of ζ zeros or a finite set of prime numbers. We calculate the Rényi dimension [10] for these potentials. The results anticipate that these potentials have multifractal nature. In Sec. III we attempt to further clarify why previous studies presented contradictory results for the fractal dimension.

II. INVERSE SCATTERING FORMALISM

Provided the Hilbert-Pólya conjecture is true, the natural and plausible approach to finding operator $\mathcal{H}$ would be to approximate it from a finite number of eigenvalues. We will

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follow this path, assuming the existence of a local potential $\mathcal{V}$ whose spectrum is related to the zeros of the Riemann $\zeta(s)$ function or, later, to the prime numbers.

The Marčhenko approach aims to reconstruct a symmetric potential using the characteristics of both the bound states (energy eigenvalues and normalization constants) and the scattering states (reflection coefficient at all energies). The question of the existence and uniqueness of any solution obtained from the inversion procedures is delicate, although if one assumes a one-dimensional, symmetric potential the complete set of eigenvalues uniquely determines the potential [11]. Different, but mathematically equivalent, methods exist [12] for reconstructing the scattering potential in a one-dimensional quantum mechanical problem.

For a given set of energy eigenvalues and reflection coefficient the quantum potential can be calculated from

$$V(x) = -2\frac{d}{dx}K(x,x')\big|_{x'=x}, \tag{2}$$

where the order of operations is important. One should localize x' first and then perform the differentiation. The function $K(x,x')$ is the solution of the Marčhenko integral equation $(x'>x)$

$$K(x,x') + K_0(x+x') + \int_x^\infty K(x,s)K_0(s+x')ds = 0, \tag{3}$$

and the kernel $K_0(z)$ is determined by the spectral parameters

$$K_0(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(k)e^{ikz}dz + \sum_{n=1}^{N} c_n e^{-\kappa_n z}. \tag{4}$$

The input data are the reflection coefficient $R(k)$ at energy $E=\hbar^2k^2/2m$, and the normalization constant c_n of the nth bound state related to the discrete energy eigenvalue $E_n=-\hbar^2\kappa_n^2/2m$.

In general, scattering states contribute to the kernel, and therefore a family of potentials can be associated with a given set of energy eigenvalues. However, according to our physical picture there are no scattering states in our case. The potential is theoretically infinitely deep, because the set of prime numbers and the set of $\zeta(s)$ zeros are infinite sets with no upper bound. Henceforward we take $R(k)\equiv 0$.

In the case of the reflectionless potential, the scattering states in Eq. (4) do not contribute to the kernel and, therefore, lead to a separable and exactly solvable integral equation. The potential can then be obtained from the formula [13]

$$V(x) = -2\frac{d^2}{dx^2}\ln[\det(\mathbf{I}+\mathbf{C})], \tag{5a}$$

where $\mathbf{I}$ denotes the identity matrix, and

$$\mathbf{C}_{m,n} = \frac{c_m c_n}{\kappa_m+\kappa_n}e^{-(\kappa_m+\kappa_n)x}, \tag{5b}$$

$$\frac{c_n^2}{2\kappa_n} = \prod_{\substack{m=1\\ m\neq n}}^{N}\left|\frac{\kappa_m+\kappa_n}{\kappa_m-\kappa_n}\right|. \tag{5c}$$

Choosing a set of eigenvalues $\{\kappa_n\}$, one may calculate the corresponding normalization constants $\{c_n\}$ and matrix $\mathbf{C}$ from Eqs. (5b) and (5c). Substituting this matrix into (5a), the desired potential can be calculated.

We note here briefly that, by using the matrix identity $\ln[\det(\mathbf{M})]=\mathrm{Tr}[\ln(\mathbf{M})]$ and the power series expansion of $\ln(1+x)$, one can symbolically derive the following expression for the potential:

$$V(x) = 2\sum_{r=1}^{\infty}\frac{(-1)^r}{r}\mathrm{Tr}\left(\frac{d^2}{dx^2}\mathbf{C}^r\right), \tag{6}$$

where $\mathrm{Tr}(\mathbf{M})$ denotes the trace of matrix $\mathbf{M}$.

The formulas (5a)–(5c) above form the basis of our calculations. Although these expressions may seem simple, the accurate numerical evaluation of the determinant can prove challenging, particularly as the number of eigenvalues is increased.

The inversion technique in its present form is numerically not convenient for more than about 500 eigenvalues, for two reasons: first, the matrices involved are dense, and, second, the numerical precision required is demanding, since for medium values of x one has to calculate the exponential functions in (5b) very accurately to have precise cancellation. The transformation of the formulas (5a)–(5c) into a numerically more tractable form is under investigation.

For this reason, for large sets of eigenvalues, we used the dressing transformation [14] to calculate the potential. We have checked numerically, up to 300 energy eigenvalues, that Marčhenko's inversion method and the dressing transformation give identical results within numerical accuracy. The strength of our approach is in the explicit formulas for the potential construction. Using the dressing transformation one has to recursively solve ordinary differential equations, since the potential is built up by incorporating the energy eigenvalues one by one, so in every step the solution of the previous step is used. Therefore, the applicability of this method to gain general and analytical results is limited. In contrast, in our method all quantities are expressed in terms of the input parameters, viz., the set of energy eigenvalues, offering some hope of analytic progress.

A. Reconstruction of well-known potentials

To illustrate the method, we reconstruct well-known potentials from their spectra, using the triangular and harmonic oscillator potentials. Later the extended Numerov method [15] is used to calculate the energy eigenvalues of the inversion potentials. This also serves to check the validity of the potentials obtained using the Marčhenko approach.

We build up the potentials using a finite number of eigenvalues, starting with one, then two, five, and finally 100 eigenvalues from the bottom of the known spectrum. As one may expect, incorporation of more and more eigenvalues into the method results in the inversion potential becoming

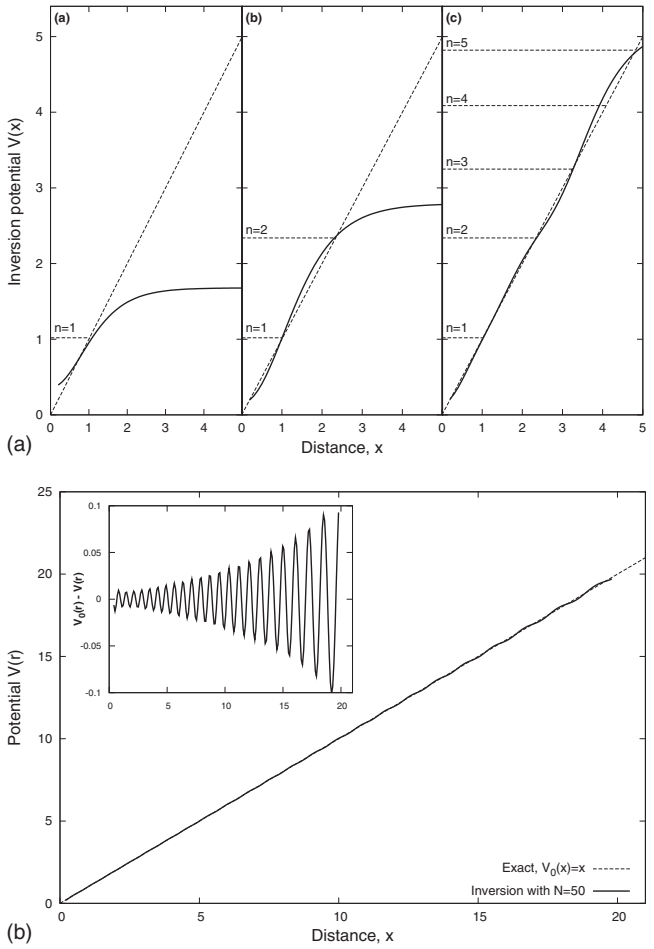

FIG. 1. Reference potential $V_0(x)=x$ (dashed line) and inversion potentials (solid lines) $V(x)$ using (a) one, (b) two, and (c) five energy eigenvalues indicated with horizontal dashed lines. The lower figure depicts the reference and inversion potentials derived using the first 50 energy eigenvalues; the inset illustrates the difference between $V_0(x)$ and $V(x)$.

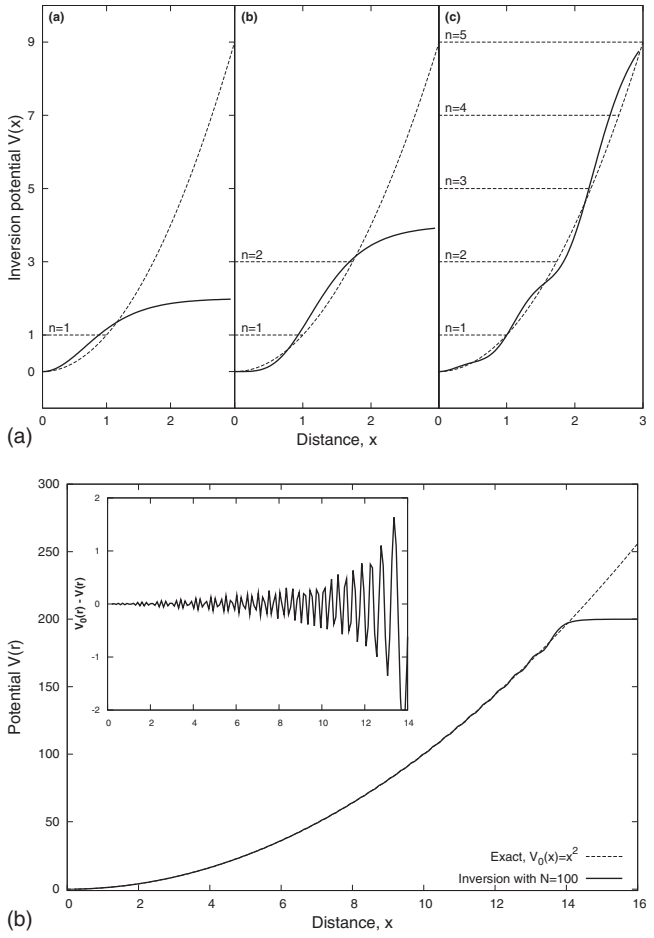

FIG. 2. Reference potential $V_0(x)=x^2$ (dashed line) and inversion potential $V(x)$ (solid line) using (a) one, (b) two, and (b) five energy eigenvalues which are indicated with horizontal dashed lines. In the lower panel, the same quantities are presented with the first 100 energy eigenvalues utilized. The inset depicts $V_0(x) - V(x)$.

more and more accurate and reproducing the spectrum faithfully. This tendency is clearly captured in Fig. 1 for the triangular potential and in Fig. 2 for the harmonic potential. Furthermore, the inversion potentials reach their asymptotic values exponentially [13] and this asymptote lies between the last energy eigenvalue used for the inversion and the next eigenvalue. Although the discrepancy between the exact and inversion potentials becomes larger toward the edge of the inversion potential, the energy eigenvalues are still correctly reproduced (see Table I) with tolerable errors.

B. Inversion potential for prime numbers

Using semiclassical arguments, one may show that for a one-dimensional potential the energy eigenvalues cannot increase more rapidly than quadratically, i.e., $\epsilon_n \sim n^2$. Intuitively this may be seen by noting that in the case of the triangular attractive potential ϵ_n scales as $n^{2/3}$, while for the harmonic oscillator ϵ_n varies as n, and, as a limiting case, for the infinite-box potential, ϵ_n varies as n^2. A corollary of the Hadamard-Poussin theorem [16] is that the nth prime number is approximately $n \ln(n)$, which is clearly less than n^2. We cannot, therefore, rule out the existence of a quantum mechanical potential which has prime numbers as energy eigenvalues.

We now turn to the construction of a semiclassical potential for which the first n energy eigenvalues coincide with the first n prime numbers. There is no theoretical limit on the number of incorporated prime numbers, although numerically the calculation becomes quite cumbersome.

Using the Wentzel-Kramers-Brillouin semiclassical quantization formula [17] and the leading terms of the prime number counting function, $\pi(E) \approx R(E)$, one may derive [18] the following implicit equation for a potential of which the eigenvalues are approximately the prime numbers ($E_0 \geq 1$):

$$x(V) = \sum_{m=1}^{\infty} \frac{\mu(m)}{m} \int_{E_0}^{V} \frac{E^{(1-m)/m}}{\ln(E)\sqrt{V-E}} dE, \tag{7}$$

where E_0 denotes the reference energy level. Due to the density of prime numbers, $\rho(E) \approx \ln(E)$, this reference energy cannot be less than 1. Even though the integral, for general

TABLE I. Comparison of the known energy eigenvalues $\epsilon_{0,n}$ for the harmonic and triangular reference potentials with the energy eigenvalues ϵ_n of the appropriate inversion potentials calculated for the first 100 eigenvalues. The table shows the first and the last five eigenvalues.

	Harmonic potential		Triangular potential	
n	$\epsilon_{0,n}$	ϵ_n	$\epsilon_{0,n}$	ϵ_n
1	1	1.001923	1.018793	1.015439
2	3	3.000020	2.338107	2.338100
3	5	5.000944	3.248198	3.247152
4	7	7.000031	4.087949	4.087942
5	9	9.000696	4.820099	4.819397
96	191	190.997520	36.995074	36.995066
97	193	192.996641	37.252699	37.252615
98	195	194.996021	37.509795	37.509785
99	197	197.015433	37.765659	37.765580
100	199	198.984293	38.021009	38.021020

m, cannot be expressed using elementary functions, one may bound the integral from below and from above such that $(x \gg 1)$

$$x^2 \ln^2(x) < V(x) - E_0 < x^2 \ln^2[x \ln(x)]. \tag{8}$$

In Fig. 3 we plot the inversion potential calculated from the first 200 prime numbers and the associated semiclassical potential from Eq. (7). It is apparent that the inversion potential oscillates around the semiclassical potential except close to the edge of the potential. Similarly, one may solve the Schrödinger equation with the semiclassical potential and with the inversion potential obtained above, comparing how well they reconstruct the original set of eigenvalues. Table II comprises a selection from the original set of eigenvalues $(\epsilon_{0,n})$, labeled by n, and the energy eigenvalues of the semiclassical $(\epsilon_{\mathrm{sc},n})$ and inversion potential (ϵ_n) with the same

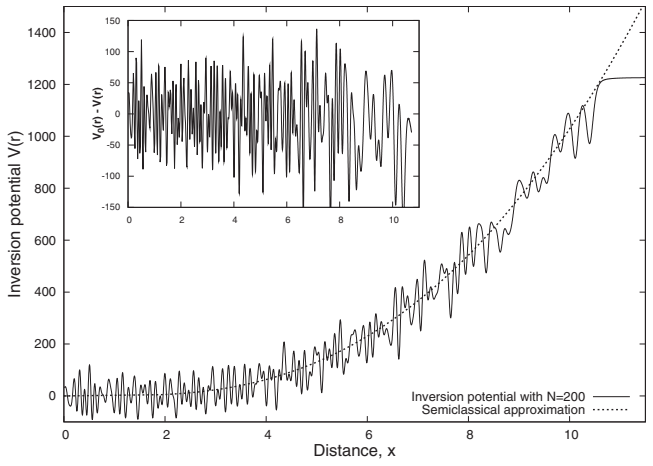

FIG. 3. Semiclassical potential (dashed line) $V_0(x)$ and inversion potential (solid line) $V(x)$, derived using Marčhenko's method with the first 200 prime numbers. Inset: Difference of $V_0(x)$ and $V(x)$.

quantum number. This also served as a numerical check of our method. The semiclassical energy eigenvalues capture the trend, but—as expected—those derived from the inversion potentials are much better approximations to the exact eigenvalues. One may notice that the agreement between $\epsilon_{0,n}$ and ϵ_n is consistently much better for even values of n. The same tendency can also be seen in Table I for the triangular and harmonic oscillator potentials, although on an order of magnitude smaller scale. The underlying reason for this effect is the subject of ongoing investigation.

C. Inversion potential for $\zeta(s)$ zeros

Similarly to the semiclassical approximation derived for the prime numbers, one may calculate a potential corresponding to the Riemann $\zeta(s)$ zeros using the fact that the number of zeros [16]

$$\mathcal{N}(E) = \frac{1}{2\pi} E \ln(E) - \frac{1 + \ln(2\pi)}{2\pi} E + \frac{7}{8} + O(\ln(E)). \tag{9}$$

Calculation of the average density of the $\zeta(s)$ zeros from the expression above restricts the choice of the otherwise arbitrary reference energy level to $E_0 \geqslant 2\pi$. Inserting the density into the Wentzel-Kramers-Brillouin semiclassical quantization formula, we obtain (see 2.727.5 in [19])

$$x(V) = \frac{1}{\pi}\left[\sqrt{V - E_0} \ln\left(\frac{E_0}{2\pi e^2} \right) + \sqrt{V} \ln\left(\frac{\sqrt{V} + \sqrt{V - E_0}}{\sqrt{V} - \sqrt{V - E_0}} \right) \right], \tag{10}$$

which is identical to that given by Wu and Sprung [20]. The structure of the semiclassical potential close to the origin depends on the choice of the reference energy level. If $E_0 > 2\pi$ then $V(x) - E_0 \sim x^2$, but in the case of $E_0 = 2\pi$ the potential grows as $V(x) - E_0 \sim x^{2/3}$. As $|x|$ approaches infinity the potential becomes independent of E_0 and expression (10) can be inverted to obtain the asymptotic

$$V(x) \approx \frac{\pi^2 x^2}{4}\left[W\left(\sqrt{\frac{\pi}{2}} \frac{|x|}{e} \right) \right]^{-2} \tag{11}$$

where $W(z)$ denotes the Lambert W function.

Applying the formulas (5a)–(5c) we calculated a number of potentials supporting the first n zeros of the $\zeta(s)$ as energy eigenvalues, utilizing a tabulated form of the low-lying zeros [21]. As an example, we have plotted the potential for $N = 200$ in Fig. 4 and in Table II one can compare how well the energy eigenvalues of the inversion potential coincide with the original eigenvalues, i.e., the zeros of the Riemann $\zeta(s)$ function. In this case the agreement is even better than it was for the prime numbers. This can be explained by the much slower increase of the potential than that for prime numbers, as x approaches infinity. Similar effects are seen for the two pedagogical examples in Table I. The energy eigenvalues for the triangular potential are, at least, an order of magnitude more accurate than those for the harmonic potential.

III. COMPARISON WITH EARLIER RESULTS

Both variational and dressing-transformation techniques have already been applied to construct quantum mechanical

TABLE II. The energy eigenvalues of the potentials derived for the prime numbers and the zeros of the $\zeta(s)$ function. For both the primes and zeros of the Riemann $\zeta(s)$ function, the first column comprises the exact eigenvalues, $\epsilon_{0,n}$, the second one contains $\epsilon_{\mathrm{sc},n}$ for the semi-classical potential, and the last incorporates the energy eigenvalues ϵ_n of the appropriate inversion potential.

	Prime numbers			Riemann $\zeta(s)$ zeros		
n	$\epsilon_{0,n}$	$\epsilon_{\mathrm{sc},n}$	ϵ_n	$\epsilon_{0,n}$	$\epsilon_{\mathrm{sc},n}$	ϵ_n
1	2	0.6895	1.6387	14.1347	13.4690	13.0302
2	3	2.5316	3.0005	21.0220	23.2274	21.0208
3	5	5.0674	4.7052	25.0109	29.8790	24.7026
4	7	7.9717	7.0006	30.4249	36.0644	30.4234
5	11	11.1201	10.702	32.9351	41.4187	32.8091
96	503	513.8440	503.0008	229.3374	284.3914	229.3354
97	509	520.3027	508.7052	231.2502	287.1530	231.2259
98	521	526.7728	520.9981	231.9872	289.9657	231.9865
99	523	533.2544	522.9371	233.6934	292.8088	233.6322
100	541	539.7472	540.9843	236.5242	295.7020	236.5215

potentials for which the energy eigenvalues are either the zeros of the Riemann $\zeta(s)$ function [14,20,22], or the prime numbers [22]. The common feature of these methods is that the potential is built up in recursion by incorporating more and more eigenvalues into the spectrum.

Previous works [20,22] estimated the box-counting dimension of the potentials belonging to the prime numbers to be 1.8, and for the Riemann $\zeta(s)$ zeros to be 1.5, where the number of eigenvalues used ranged from 100 to 32 000. Our measurements broadly support these values (see D_0 in Fig. 5).

In order to reproduce these findings we treat the graph of the potential as a signal. To measure the fractal dimension we detrend the signal, i.e., subtract the actual inversion potential from the semiclassical potential, $\xi(x)=V(x)-V_{\mathrm{sc}}(x)$. Moreover, we limit ourselves to the spatial range of [0,10] to eliminate any boundary effect arising from the fact that both

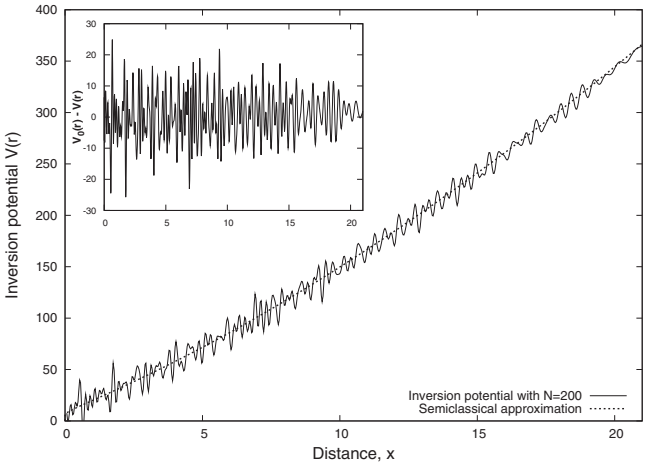

FIG. 4. Semiclassical potential (dashed line) $V_0(x)$ and inversion potential (solid line) $V(x)$, derived from the inverse scattering method using the first 200 energy eigenvalues. Inset: Difference of $V_0(x)$ and $V(x)$.

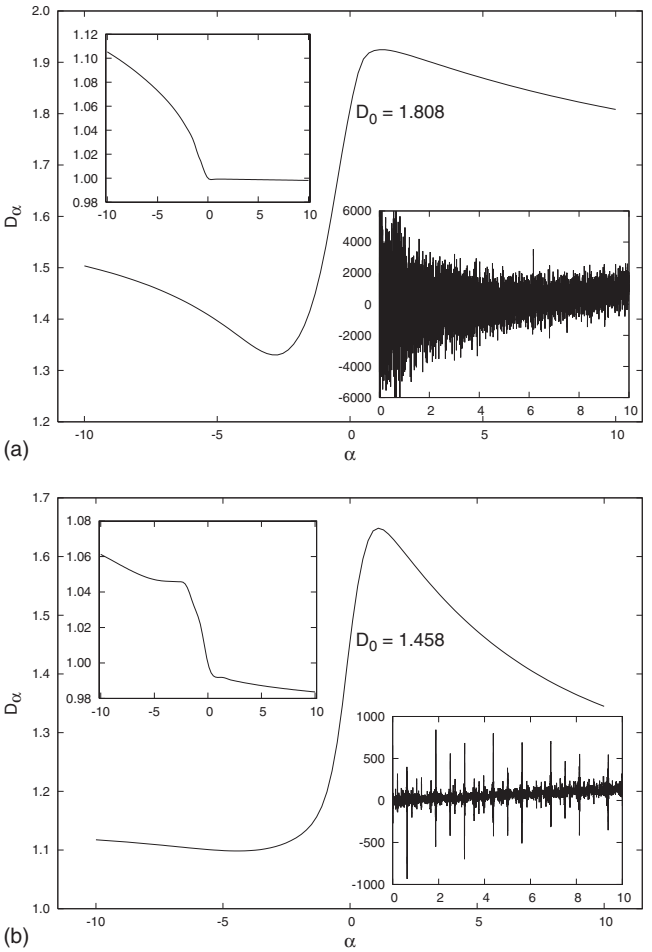

FIG. 5. Rényi dimension D_α shown for the potentials belonging to (a) the prime numbers and (b) the Riemann $\zeta(s)$ zeros. Insets on the left show D_α for the marginal distribution (see text) while those on the right depict the detrended data. Here, in the dressing transformation, we used the first 10^5 eigenvalues and spatial step size $h=10^{-6}$ for both the prime numbers and the zeros of $\zeta(s)$.

Marčhenko's method and the dressing transformation produce a potential with a constant asymptotic value for large spatial coordinates.

We have measured the Rényi dimension of the potentials [10,23], defined as

$$D_\alpha(X) = \frac{1}{\alpha - 1} \lim_{\epsilon \to 0^+} \left(\frac{\ln\left(\sum_{i=1}^{N} p_i^\alpha \right)}{\ln(\epsilon)} \right), \tag{12}$$

where p_i is the probability that the discrete random variable X falls into a box centered at x_i with side ϵ. This probability can be approximated using the relative frequencies obtained by dividing the embedding two-dimensional (x, V) space into a finite number of bins and counted how many times the potential takes its value in the given box. In contrast to the general box-counting method, which treats every box equally, regardless of how many points of the fractal a given box contains, if $\alpha > 0$ in (12) then boxes with higher relative frequencies will dominate the summand, therefore determining $D_\alpha(X)$. On the other hand, if $\alpha < 0$ then the formula weighs with less dense boxes more and measures their scaling properties. As a special case, $\alpha = 0$ associates equal weights with every box, and therefore D_0 should reproduce the box-counting dimension. It can also be shown [10] that for $\alpha \to 1$ the numerator including the prefactor converges to the Shannon entropy, defined to be $-\Sigma_i p_i \ln(p_i)$. Concluding, by calculating the generalized Rényi dimension one can "scan" the fractal by its "density," and measure its heterogeneity. If D_α varies with α then the fractal is a multifractal, since its subsets scale differently.

One may derive two statistics for ξ based on (12): (a) using a two-dimensional grid and approximating the two-dimensional conditional probability distribution with the relative frequencies of the individual boxes, or (b) calculating the generalized dimension for the marginal probability distribution of ξ. In Fig. 5 both sets of statistics are demonstrated, showing the generalized dimension for the two-dimensional probability distribution and the insets depicting D_α for the marginal distribution. The box-counting dimensions D_0 are also indicated on the graphs. Although both curves in the main figures have an overall S shape, their structures are different. In the case of the potential generated from the prime numbers, both the positive and negative α branches of the curve show a monotonic decrease toward the right. However, for the potential designed from the zeros of the Riemann $\zeta(s)$ function, the negative α branch of the curve remains nearly constant.

These results suggest that the potentials calculated for the prime numbers and for the zeros of the Riemann $\zeta(s)$ function are indeed multifractals [24]. The steep middle part of the curves also explains why earlier studies [14,20,22,25] differed in the box dimension. The number of incorporated energy eigenvalues strongly influences the conditional probability associated with one box and eventually shifts D_0.

Finally, we mention another conjectured property of the quantum system supposed to possess the zeros of the Riemann $\zeta(s)$ function as energy eigenvalues, namely, that $\mathcal{H}$ may violate time-reversal symmetry.

Similarly to the prime counting function $\pi(x)$, one may define a function $\mathcal{N}(t)$ that counts the zeros of the Riemann $\zeta(s)$ function, i.e., a function jumping by unity whenever t passes over one of the zeros, t_n. It has been proved [16] that the function $\mathcal{N}$ can be decomposed into a smooth and a fluctuating part, $\mathcal{N}(t) = N(t) + \mathcal{N}_{\mathrm{osc}}(t)$, where $N(t)$ has been given in (9) explicitly. The fluctuating term has a remarkably similar structure to Gutzwiller's trace formula [26] giving the density of states of a quantum system. The comparison of the two formulas led to the hypothesis [27] that a quantum system with the zeros of the Riemann $\zeta(s)$ function as energy eigenvalues does not possess time-reversal symmetry. The approach presented in this paper creates a symmetric, one-dimensional, although multifractal, potential $V(x)$, for which the corresponding energy eigenvalues coincide with the first n nontrivial zeros of the Riemann $\zeta(s)$ function. The reflection symmetry of the potential, $V(x) = V(-x)$, guarantees time-reversal symmetry of the Hamiltonian, $\mathcal{H} = p^2/2m + V(x)$. This result, therefore, allows us to assume the existence of a quantum system having the ζ zeros as energy eigenvalues and obeying time-reversal symmetry simultaneously.

IV. CONCLUSION

In the present paper we used Marčhenko's method, one of the inverse scattering methods, to construct one-dimensional, symmetric quantum potentials, the energy eigenvalues of which coincide with either the prime numbers or the zeros of the Riemann $\zeta(s)$ function. We have demonstrated the accuracy and usefulness of this method on two pedagogical examples, the triangular and harmonic potentials. For both cases we showed the reconstructed potentials and calculated the energy eigenvalues, which agreed with the predescribed values very well. Later, we applied the same technique and numerically calculated potentials for the prime numbers and zeros of $\zeta(s)$. We found that the outcome of the Marčhenko method is identical to that of the dressing transform used previously. At the present stage, the latter method is numerically preferable to Marčhenko's method. Using the dressing transform, we created potentials, to high accuracy, from the first 100 000 prime numbers and also for the same number of zeros of the Riemann $\zeta(s)$ function. Looking at the graphs of these potentials as signals, one can analyze their statistical properties. After detrending these signals we calculated the Rényi dimension, which is a generalized fractal dimension. Our results suggest that inversion potentials are multifractals for both the prime numbers and the zeros of $\zeta(s)$. The specific values of the generalized dimension for the prime numbers $D_0 = 1.808$, and for the $\zeta(s)$ zeros, $D_0 = 1.458$, agree well with [22].

Even though Marčhenko's method is not yet able to compete with the dressing transform in the number of eigenvalues incorporated into the potential, it gives explicit formulas for how one can build up such potentials without recursion. Looking at formula (5a) one may see that the determinant of

the matrix $\mathbf{I}+\mathbf{C}$ is a polynomial of the entries, i.e., of exponential functions. Taking the natural logarithm and differentiating twice with respect to the spatial variable will not change the fact that the potential is a rational function of a finite number of exponentials. This fact is more apparent in formula (6). This gives us the hope of being able to explore analytically the properties of a quantum system which possess the zeros of the Riemann $\zeta(s)$ function as energy eigenvalues.

Simplification of formulas (5a)–(5c) is under investigation.

ACKNOWLEDGMENTS

D.S. is grateful to Dr. Dennis McCaughan for the discussion about the convergence of a sequence containing the Möbius function. This work was financially supported by the New Zealand Foundation for Research Science and Technology under Contract No. NERF-UOOX0703: Quantum Technologies. B.P.v.Z. acknowledges financial support from the Natural Sciences and Engineering Research Council (NSERC) of Canada.

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